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Computer Science: information system, information retrieval system, concept language.

Abstract:
A model of a collection of documents based on *partial Boolean algebras* is presented. This model has been considered while analysing a problem of *integration of thesauri*. Some properties of partial Boolean algebras are exploited in defining theoretical tools for information retrieval associated to this model. Such tools are a logical language representing queries to the system and a browsing mechanism. In the appendix, a simple example of such an integration process is reported.

## 1. INTRODUCTION

Thesaurus integration is a branch of Library Science; it aims at the definition of a technique for integrating various indexes of a *collection* of documents into a single — and coherent — tool for indexing and retrieving [J. Aitchison, A. Gilchrist, D. Bawden, 1997].

In this paper, a very simple definition of thesaurus is considered; in fact we assume that each thesaurus represents a *perspective* on the collection of documents and we assume that the relations among the index words in the thesaurus are simply given by set containment among the respective indexed sets of the collection. Moreover, we assume that each thesaurus is built over a plain set of index terms, a *facet*, representing a *partition* of the collection of documents.

We assume as given the following objects: a *finite* set $X$ representing the collection of documents (each element in $X$ being a document); an *index language* $L$ whose index terms come from a finite set $\mathcal{F}$ of finite facets; each index term $l$ in $L$ is associated — by an *extent* operator $\mathcal{E}: L \to \wp(X)$ ($l \overset{\mathcal{E}}{\mapsto} x$) — to a *subset* $x$ of $X$. The set $x$ is called *extent* of $l$; $l$ is called the *intent* of $x$ and the couple $\langle x, l\rangle$ is called *concept* [B. A. Davey, H. A. Priestley, 1990]. Moreover, we assume that the extents of each *facet* $F = \{l_1, l_2, \ldots, l_n\} \in \mathcal{F}$ are a partition of $X$ ($\bigcup_{i=1,\ldots n} \mathcal{E}(l_i) = X$ and $\forall i, j \; i \neq j \to \mathcal{E}(l_i) \cap \mathcal{E}(l_j) = \emptyset$). By $\mathcal{E}(F)$ we mean the sets composing the partition of $X$ induced by the facet $F$.

For each facet $F$ we can build a boolean algebra by considering all the unions of elements in $\mathcal{E}(F)$. The elements in $\mathcal{E}(F)$ are the atoms of such a Boolean algebra, $\emptyset$ is the zero element and the whole collection $X$ is the 1 element. On the 'intent' side, we have the index terms $l_1, l_2, \ldots, l_n$ as 'atomic intents', while any combination $\alpha$ of them via the usual logical connectives — $a \rightarrow b = \neg a \vee b$ — will produce a term whose extent is defined by applying recursively $\mathcal{E}(\,.\,)$ to the elements of $\alpha$ and composing them by the Boolean operators corresponding to the logical connectives. The elements $\alpha$ — possibly rewritten in a more intuitive form — are the index terms of the thesaurus associated to the given facet.

In the general case we have various facets, and from each one of them we are able to build the associated thesaurus and Boolean algebra. What we want to do now, is to 'glue' coherently the various thesauri while preserving the particular perspective in which each of them sees the collection $X$.

## 2. ALGEBRAIC REPRESENTATION AND QUANTUM LOGICS

We have now a set $\mathcal{B}_{\mathcal{F}}$ of boolean algebras whose index set $\mathcal{F}$ is the set of the facets. We assume, for $f$ and $g$ elements of two distinct facets, that $\mathcal{E}(f) = \mathcal{E}(g) \Rightarrow f = g$. So both the intents and the extents of our collection of Boolean algebras will have elements in common — at least the elements 0 and 1 will be in common to all the algebras — and equal extents constrain equal intents.

A structure $\mathcal{B}$ is called a *partial Boolean algebra* (PBA) if some conditions on the common elements hold:

Definition 1. *Partial Boolean algebra (PBA).* Let $\mathcal{B}_I$ be a family of Boolean algebras indexed by a set $I$. Let each $B_i \in \mathcal{B}_I$ be defined as: $B_i = \langle B_i, \wedge_i, \vee_i, (\,.\,)'_i, 0_i, 1_i \rangle$ — $(\,.\,)'_i$ being the operation of complement in the $i^{th}$ Boolean component.

$$\mathcal{B} = \langle \bigcup_{i \in I} B_i, \wedge, \vee, (\,.\,)', 0, 1 \rangle$$

is a *partial Boolean algebra* if the following conditions hold for all $i, j \in I$, and for all $x, y \in B_i \cap B_j$:

$$B_i \cap B_j = B_k \text{ for some } k \in I;$$
$$0_i = 0_j \text{ and } 1_i = 1_j;$$
$$x \vee_i y = x \vee_j y \text{ and } (x)'_i = (x)'_j.$$

Moreover, for all $i, j, k \in I$, a *coherence condition* holds:

$$x, y \in B_i \text{ and } y, z \in B_j \text{ and } z, x \in B_k \rightarrow \exists h \in I \text{ such that } x, y, z \in B_h.$$

$\wedge, \vee$ and $(\,.\,)'$ are defined as the union of the corresponding operators $\wedge_i, \vee_i, (\,.\,)'_i$; $0 = 0_i$ and $1 = 1_i$ for some $i$. In the general case $\wedge$ and $\vee$ are *partial* operations.

Given a PBA $\mathcal{B}$, it is possible to define a relation $\leq$ on $\mathcal{B}$ as (for all $x, y \in \mathcal{B}$): $x \leq y$ if, and only if, $x \wedge y = x$; a PBA $\mathcal{B}$ is said to be *transitive* if $\leq$ is transitive. Two elements, $x$ and $y$ in $\mathcal{B}$ are said to be *compatible* (noted by $x \,\$\, y$) if and only if they belong to the same Boolean sub-algebra:

$$x \,\$\, y \;\Leftrightarrow\; \exists i \in I\,(x, y \in B_i).$$

In the general case, the collection of the Boolean algebras built over the set of facets $\mathcal{F}$ does not define a PBA. In order to have a PBA, it is necessay to 'saturate' $\mathcal{F}$ with other facets, implicitly defined by the facets already present in $\mathcal{F}$. This 'saturation' is required, for example, by the fact that each element $y$ — together with its complement $y'$, the whole $X$ and $\emptyset$ — forms a Boolean sub-algebra that has to be included in $\mathcal{B}$; another — more complex — situation arises when dealing with the coherence condition. It is possible to define a (terminating) process of construction of this 'saturated' set of facets and to prove that it results in a transitive PBA [C. Ferigato, 1998]. This 'saturated' set of facets will be noted by $\hat{\mathcal{F}}$. The Boolean algebras $B_F$ obtained as above from the saturated set of facets $\hat{\mathcal{F}}$ will form a transitive partial Boolean algebra $\mathcal{B}$:

$$\mathcal{B} = \langle \bigcup_{F \in \hat{\mathcal{F}}} B_F, \wedge, \vee, (\,.\,)', \emptyset, X \rangle;$$

where the operations $\wedge$ and $\vee$ are defined in terms of $\cap$ and $\cup$ in the Boolean components and result in partial operations; $(\,.\,)'$ is the set complement $(X \setminus (\,.\,))$. In what follows, $\mathcal{B}$ will denote both the PBA and its support set $\bigcup_{F \in \hat{\mathcal{F}}} B_F$.

Partial Boolean algebras are known among the algebraic models of quantum logic; in this particular case, we are using a partial Boolean algebra equivalent to an orthomodular poset [P. Pták, S. Pulmannová (1991)], [R. I. G. Hughes, (1989)].

## 3. FILTERS AS MAXIMAL AND CONSISTENT SETS OF PROPOSITIONS

We now associate a logical language $\mathcal{L}$ to the PBA $\mathcal{B}$. The terms of this language come from the 'saturated' set of facets $\hat{\mathcal{F}}$ above. Let $\hat{L}$ be the set of the equivalence classes — with respect to the extent operator $\mathcal{E}(\,.\,)$ — of the index terms $l_i$ of all the facets in $\hat{\mathcal{F}}$ ($l_1 = l_2 \;\Leftrightarrow\; \mathcal{E}(l_1) = \mathcal{E}(l_2)$); $\hat{L}$ represents, intuitively, all the index terms of the thesaurus resulted from the 'gluing' operation presented in the introduction. Then, the language $\mathcal{L}$ is defined as:

$$
\begin{aligned}
a \in \hat{L} &\Rightarrow a \in \mathcal{L};\\
a \in \mathcal{L} &\Rightarrow \neg a \in \mathcal{L};\\
a, b \in \mathcal{L} &\Rightarrow a \wedge b,\ a \vee b,\ a \rightarrow b \in \mathcal{L};\\
&\textit{no other formula is in } \mathcal{L}.
\end{aligned}
$$

Not all the formulas in $\mathcal{L}$ have a corresponding element in $\mathcal{B}$, since the application of $\mathcal{E}(\,.\,)$ to arbitrary formulas in $\mathcal{L}$ depends from the partial operations $\wedge$ and $\vee$ in $\mathcal{B}$. Anyway, all the elements in $\hat{L}$ have a corresponding element in $\mathcal{B}$ since, by the construction process that lead to $\hat{F}$, their extents are all the non zero elements of $\mathcal{B}$.

In this section we are interested in finding sets of propositions of $\mathcal{L}$ that are maximal and not contradictory. The proofs of all the properties used in this section are reported in: [C. Ferigato, 1998] and in [L. Bernardinello, C. Ferigato, L. Pomello, 1998]. This latter paper contains a more general discussion on filters in partial Boolean algebras — and orthomodular posets — in the context of distributed systems theory.

In Boolean algebras, maximal and consistent sets of propositions are identified by *ultrafilters*. The analogue of an ultrafilter in finite transitive partial Boolean algebras is a *prime filter*.

Definition 2. *Filter.* Let $\mathcal{B}$ be a transitive partial Boolean algebra, $A \subseteq \mathcal{B}$ is a *filter* in $\mathcal{B}$ if:

$$
\begin{aligned}
&A \neq \emptyset;\\
&x \in A \text{ and } x \leq y \Rightarrow y \in A;\\
&x, y \in A \text{ and } x \;\$\; y \Rightarrow x \wedge y \in A.
\end{aligned}
$$

A filter $A$ is *proper* if $A \neq \mathcal{B}$; $A$ is *prime* if ($\forall x, y \in \mathcal{B}$):

$$x \;\$\; y \text{ and } x \vee y \in A \Rightarrow x \in A \text{ or } y \in A.$$

If $A$ is a prime filter in $\mathcal{B}$, it is possible to prove that $x \in A \;\Leftrightarrow\; x' \notin A$; the set of all prime filters in $\mathcal{B}$ will be denoted by $PF_{\mathcal{B}}$.

The partial Boolean algebras we are dealing with have the property of having 'enough' prime filters for separating all the elements of the algebra itself. Formally, such a partial Boolean algebra is called *prime*:

Definition 3. *Prime partial Boolean algebra.* A transitive PBA $\mathcal{B}$ is *prime* if, and only if, $\forall x, y \in \mathcal{B} : x \neq y \Rightarrow \exists A \in PF_{\mathcal{B}}$ such that $x \in A \;\Leftrightarrow\; y \notin A$.

Since prime filters in PBAs are analogue to ultrafilters in finite Boolean algebras, it is possible to compute prime filters in (finite, prime and transitive) PBAs as up-closures of particular subsets of its atoms. As expressed in the theorem below, it is sufficient to choose exactly one atom from each maximal Boolean subalgebra composing $\mathcal{B}$. In what follows, $\not\$$ is defined as:

$$\langle x, y \rangle \in \not\$ \;\Leftrightarrow\; \langle x, y \rangle \notin \$,$$

and $\varphi\uparrow$ represents the up-closure of some subset $\varphi$ in $\mathcal{B}$.

<u>Theorem</u> 1. *Maxcliques.* Let $AT_{\mathcal{B}}$ be the set of atoms of a (finite, prime and transitive) PBA $\mathcal{B}$, let $\mathcal{S}$ be the set of the maximal cliques of $^{\$}$ in $AT_{\mathcal{B}}$, let $\mathit{\$}$ be the set of the maximal cliques of $^{\not\$}$ in $AT_{\mathcal{B}}$. Let $\mathcal{C}$ be defined as:

$$\mathcal{C} = \{\varphi \in \mathit{\$} \mid \forall K \in \mathcal{S}\ (|\varphi \cap K| = 1)\},$$

let $PF_{\mathcal{B}}$ be the set of all prime filters in $\mathcal{B}$, then

$$\varphi\uparrow \in PF_{\mathcal{B}} \ \Leftrightarrow\ \varphi \in \mathcal{C}.$$

The elements of $\mathcal{S}$ in this theorem are the sets of all the atoms of each maximal Boolean subalgebra composing $\mathcal{B}$. The elements of $\mathit{\$}$ are sets of atoms too, but no couple of them belongs to the same maximal Boolean subalgebra of $\mathcal{B}$. The definition of $\mathcal{C}$ above requires that exactly one atom from each maximal Boolean subalgebra in $\mathcal{B}$ be chosen in constructing $\varphi$.

With the teoretical background provided by the theorem above, we can define a new model and a sound interpretation function for the formulas of the language $\mathcal{L}$. What we obtain in this new model is the fact that the evaluation function is a total function. If we had limited ourselves to the PBA $\mathcal{B}$ model, the evaluation function would have been partial. A model $\mathcal{M} = \langle W, R, V\rangle$ for the language $\mathcal{L}$ is defined on the basis of the cliques $\mathcal{C}$ of $^{\not\$}$ as in the theorem above. The elements of $W$ are called *worlds*, $R$ is called *accessibility relation* and $V$ is an evaluation function assigning a truth value to each couple formula-world.

<u>Definition</u> 4. *Model*

$$\begin{aligned} &W = \mathcal{C};\\ &R \subseteq W \times W\\ &V : \mathcal{L} \times W \to \{\text{True}, \text{False}\}.\end{aligned}$$

The evaluation function $V$ can be defined recursively as follows (let $a$ be a formula in $\mathcal{L}$ and $\varphi, \psi$ be worlds in $W$):

$$\begin{aligned} a \in \hat{L} \Rightarrow (a,\varphi) &\overset{V}{\mapsto} \text{True} \quad \Leftrightarrow \quad \mathcal{E}(a) \in \varphi\uparrow;\\ a = b \vee c \Rightarrow (a,\varphi) &\overset{V}{\mapsto} \text{True} \quad \Leftrightarrow \quad \text{either } V(b,\varphi) = \text{True or } V(c,\varphi) = \text{True};\\ a = b \wedge c \Rightarrow (a,\varphi) &\overset{V}{\mapsto} \text{True} \quad \Leftrightarrow \quad V(b,\varphi) = \text{True and } V(c,\varphi) = \text{True};\\ a = b \to c \Rightarrow (a,\varphi) &\overset{V}{\mapsto} \text{True} \quad \Leftrightarrow \quad \text{either } V(b,\varphi) = \text{False or } V(c,\varphi) = \text{True};\\ a = \neg b \Rightarrow (a,\varphi) &\overset{V}{\mapsto} \text{True} \quad \Leftrightarrow \quad V(b,\varphi) = \text{False}.\end{aligned}$$

What remains to be defined is the accessibility relation $R$ and the extension of $\mathcal{L}$ to the modal connectives $\Box$ and $\Diamond$. Before defining them, we present (in the next section) an intuitive justification for these logical connectives and of their interpretation.

## 4. BROWSING THE COLLECTION

We have a modal model $\langle\mathcal{C}, R, V\rangle$ for the logical language $\mathcal{L}$; a proposition $a$ in $\mathcal{L}$ evaluated to be true in $\langle\mathcal{C}, R, V\rangle$ with respect to the world $\varphi$ is denoted by $\langle\mathcal{C}, R, V\rangle \models_{\varphi} a$. By using the maxclique theorem above, we can see $\varphi$ as a maximal and consistent set of propositions on the retrieval system modelled by $\mathcal{B}$, a *state* of the system; the Boolean components can be seen as sub-spaces of the retrieval system, while the original facets in $\mathcal{F}$ can be seen as coordinated axes.

Any two states $\varphi$ and $\psi$ of the system are not at the same distance, since the number of coordinates (atoms) that have to change while passing from $\varphi$ to $\psi$ is not always the same. By exploiting this fact, we can give a simple definition of the relative distance between $\varphi$ and $\psi$ by means of the set operation of symmetric difference.

Definition 5. *Distance, d.* Let $\varphi, \psi$ be two cliques in $\mathcal{C}$, then

$$d(\varphi, \psi) = |(\varphi \setminus \psi) \cup (\psi \setminus \varphi)|.$$

We can now define the set of the cliques at a given distance $d$ from $\varphi$:

Definition 6. $\varphi_{\approx}$. Let $\varphi \in \mathcal{C}$, let $S(\varphi, d)$ be the set of elements $\psi$ in $\mathcal{C}$ whose distance from $\varphi$ is less or equal to $d$ (the sphere of radius $d$ centered in $\varphi$); then $\varphi_{\approx}$ is the minimum nonempty sphere centered in $\varphi$ and containing at least one element different from $\varphi$.

By using $\varphi_{\approx}$, we can now define a relation of 'proximity' between cliques:

Definition 7. $\approx$. Two cliques $\varphi$ and $\psi$ of $\$$ in $\mathcal{C}$ belong to the relation $\approx$ if, and only if, $\varphi \in \psi_{\approx}$ or $\psi \in \varphi_{\approx}$.

The relation $\approx$ above is reflexive and symmetric, but not transitive.

The relation $\approx$, can be used in designing a browsing mechanism for the collection $X$ of documents in the following way: once a state $\varphi$ of the system has been selected, a new state $\psi$ *close* to $\varphi$ with respect to $\approx$ can be computed. In this way 'similar' documents can be presented to the user, this 'similarity' between cliques (or states) is directly related to the number of atoms that both have in common.

The relation $\approx$ is also the intuitive justification to the extension of the logical language $\mathcal{L}$ and of its evaluation funcion defined above by considering the modal connectives $\Box$ and $\Diamond$. The complete definition of $\mathcal{L}$ and $V$ is as follows:

Definition 8. *Language $\mathcal{L}$.*

$$\begin{gathered} a \in \hat{L} \Rightarrow a \in \mathcal{L}; \\ a \in \mathcal{L} \Rightarrow \neg a,\ \Diamond a,\ \Box a \in \mathcal{L}; \\ a, b \in \mathcal{L} \Rightarrow a \wedge b,\ a \vee b,\ a \to b \in \mathcal{L}; \\ \textit{no other formula is in } \mathcal{L}. \end{gathered}$$

Definition 9. *Evaluation function.* Let $\mathcal{C}$ be the subset of the maximal cliques of $\$$ in $\mathcal{B}$ as defined in theorem maxcliques. Let $\approx$ be the relation of proximity defined above, then a modal model $\mathcal{M} = \langle \mathcal{C}, \approx, V \rangle$ for the language $\mathcal{L}$ is defined with $V$ as follows:

$$\begin{aligned} a \in \hat{L} \Rightarrow (a, \varphi) &\overset{V}{\mapsto} \text{True} &&\Leftrightarrow \quad \mathcal{E}(a) \in \varphi\uparrow; \\ a = b \vee c \Rightarrow (a, \varphi) &\overset{V}{\mapsto} \text{True} &&\Leftrightarrow \quad \text{either } V(b, \varphi) = \text{True} \text{ or } V(c, \varphi) = \text{True}; \\ a = b \wedge c \Rightarrow (a, \varphi) &\overset{V}{\mapsto} \text{True} &&\Leftrightarrow \quad V(b, \varphi) = \text{True} \text{ and } V(c, \varphi) = \text{True}; \\ a = b \to c \Rightarrow (a, \varphi) &\overset{V}{\mapsto} \text{True} &&\Leftrightarrow \quad \text{either } V(b, \varphi) = \text{False} \text{ or } V(c, \varphi) = \text{True}; \\ a = \neg b \Rightarrow (a, \varphi) &\overset{V}{\mapsto} \text{True} &&\Leftrightarrow \quad V(b, \varphi) = \text{False}; \\ a = \Box b \Rightarrow (a, \varphi) &\overset{V}{\mapsto} \text{True} &&\Leftrightarrow \quad \forall \psi \in W\ (\varphi \approx \psi \Rightarrow V(b, \psi) = \text{True}); \\ a = \Diamond b \Rightarrow (a, \varphi) &\overset{V}{\mapsto} \text{True} &&\Leftrightarrow \quad \exists \psi \in W\ (\varphi \approx \psi \Rightarrow V(b, \psi) = \text{True}). \end{aligned}$$

What results is a modal logical system **B** (or KTB) model in the classification provided in: [G. E. Hughes, M. C. Cresswell (1984)].

## 5. CONCLUSION

Throughout this work we presented our ideas on the algebraic integration of various thesauri, each of them representing particular ordering *perspective* on a collection of documents. This is only a first step towards an algebraic theory of thesaurus integration. The technique for integration proposed preserves the original perspectives and the original index languages (this would have not been the case, if a Boolean algebra had been constructed on the intersections of all the extents of the original facets).

Moreover, the proposed model allows for the construction of a logical system representing the collection and of a browsing mechanism.

Further work has to be done in various directions: firstly in implementing a retrieval system based on the presented theory; secondly in comparing the proposed model with other existing logical and algebraic models; thirdly in exploring the logical properties of the presented model. The first further direction of work aims at evaluating the effectiveness of a retrieval system implemented with respect to the theory presented here; the second further direction of work aims at placing the presented theory in its proper position in the Information Retrieval research field, by considering, for example, the algebraic theory of the *information systems* — originated by [Z. Pavlak, 1973] — the logical theory of Information Retrieval — originated by [C. J. van Rijsbergen (1986)] — and the unified definition of the classical models. Moreover, the relations with the presented model and the use of quantum logical principles in Information Retrieval have to be compared with other works already published on this subject, for example [C. J. van Rijsbergen (1996)] and [D. Sándor (1994)].

## 6. ACKNOWLEDGMENTS

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## 8. APPENDIX

In this appendix, a simple example of the integration process proposed in the previous pages is presented.

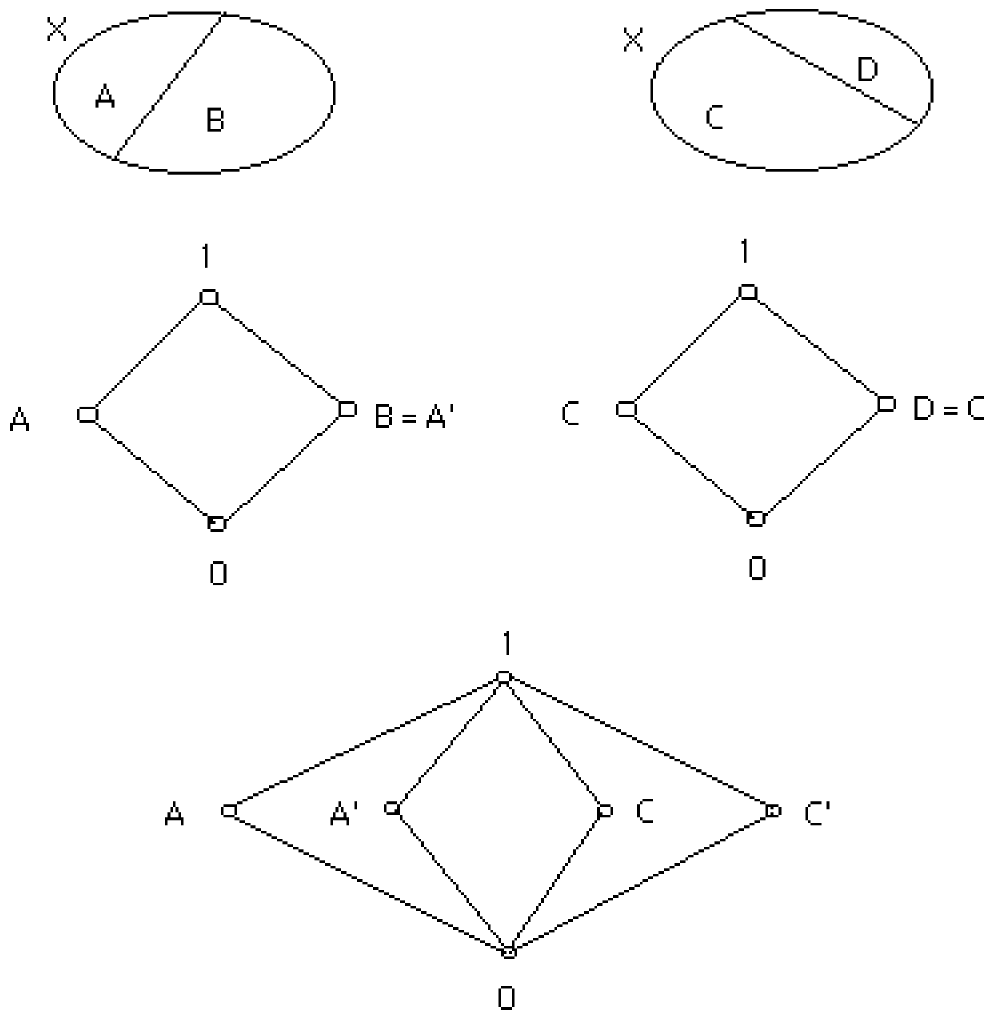


Fig. 1

Figure 1 represents two facets on the same collection $X$ of documents. These facets are used in building the two Boolean algebras $\mathcal{A}_1 = \{\emptyset = 0, A, A' = B, 1 = X\}$ and $\mathcal{A}_2 = \{\emptyset = 0, C, C' = D, 1 = X\}$. These algebras, in turn, are used in building the partial Boolean algebra $\mathcal{B}$ drawn in the lower part of the figure; the elements of $\mathcal{B}$ are $\{0, A, A', C, C', 1\}$. The operators $\wedge$ and $\vee$ — inheridted by $\mathcal{B}$ from the Boolean components $\mathcal{A}_1$ and $A_2$ — are partial; for example, $A \wedge C$ is not defined in $\mathcal{B}$, since $A$ and $C$ belong to two distinct Boolean components glued only on the elements 0 and 1.

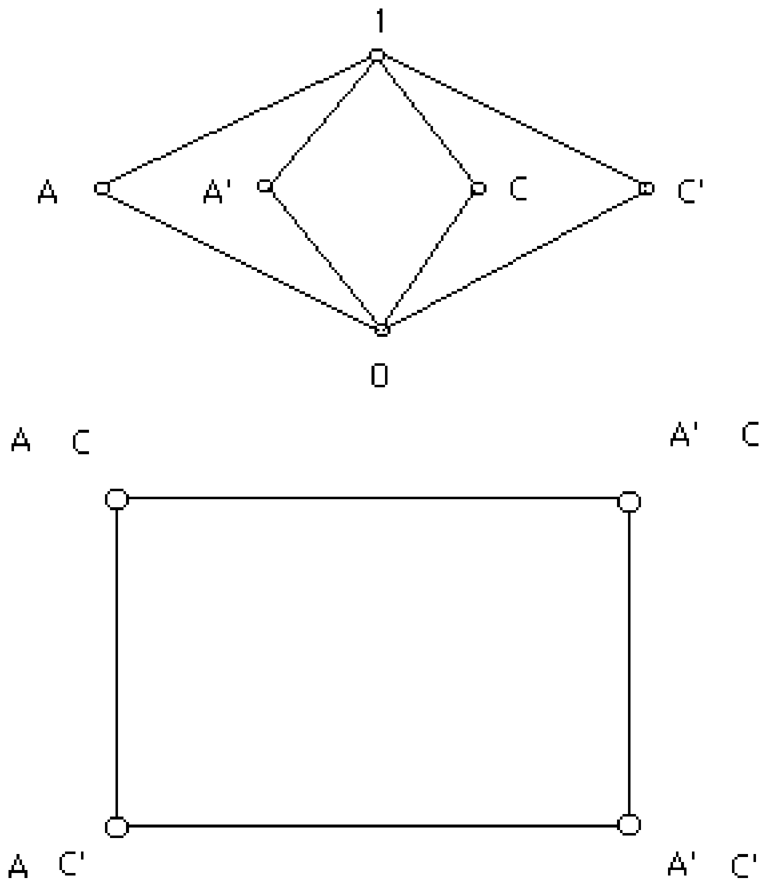


Fig. 2

Figure 2 represents the partial Boolean algebra $\mathcal{B}$ and its four prime filters. These prime filters are the up-closures of the cliques $\mathcal{C}$ of the relation $\not\$$ restricted to the atoms of $\mathcal{B}$ as in theorem 1 ($\varphi_1 = \{A, C\}$ implies $\varphi_1 \uparrow = \{A, C, 1\}$). As a consequence of theorem 1, the four filters are represented by the cliques $\varphi_1 = \{A, C\}$, $\varphi_2 = \{A, C'\}$, $\varphi_3 = \{A', C'\}$, and $\varphi_4 = \{A', C\}$. $\varphi_1, \ldots, \varphi_4$ are also the worlds of the logical model of definition 4; the accessibility relation of this model — the relation $\approx$ of definition 7 — is represented here by the arcs connecting $\{A, C\}$, $\{A, C'\}$, $\{A', C'\}$, $\{A', C\}$ in the lower part of the figure (arcs corresponding to the reflexive property are omitted).