„Golden” Properties of the World Wide Web

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Abstract

The experimental discovery that the degree distribution for Web pages and Internet nodes follows a power law was a basic milestone towards the emergence of a new science of the Web. The formulation of the principle of preferential attachment triggered research into trying to explain why the Web link topology evolves according to a power law. There are not any further results as regards properties if the power law governing the behaviour of the Web link topology. In the present paper, based on experimental evidence obtained thus far, interesting and useful new properties of the Web power law are established and discussed as follows: (i) a concept of a ‘beauty’ in the Web can be identified in that the Web power law can be characterised by the Golden Section; (ii) a numeric procedure is given, based on Fibonacci and Lucas numbers, to writing the power law for an actual Web portion.

1 INTRODUCTION

The experimental discovery by Faloutsos et al. (1999) that the degree distribution for Web pages and Internet nodes follows a power law was a basic milestone towards the emergence of a new science of the Web. The formulation of the principle of preferential attachment (Barabási et al., 2000) triggered research into and stimulated ideas towards trying to explain why the Web link topology evolves according to a power law. Pennock et al. (2002) showed that this principle is not necessarily valid for the real Web, and they proposed a modified principle that better explains the development of a power law in the real Web.

To the best of our knowledge there are not any further results as regards properties of the power law governing the behaviour of the Web link topology. The aim of the present paper is just this. Interesting and useful properties of the Web power law are established, discussed and supported by experimental evidence.

2 POWER LAW

Given a discrete random variable $V = (v_i > 0)_{i=1,\dots,n}$. If the probability P that the random variable V takes on values equal to or greater than some value v, i.e., $P(V \geq v)$, is given by

$$P(V \geq v) = \left(\frac{m}{v}\right)^k, \quad (1)$$

where $m > 0$ and $k > 0$ are — problem-dependent — constants, and $v \geq m$, then we say that the random variable V follows Pareto's Law. For example, persons' incomes obey Pareto's Law (Guilmi et al., 2003); m represents a minimal income. It follows from (1) that:

$$P(V < v) = 1 - \left(\frac{m}{v}\right)^k, \quad (2)$$

which is the distribution function $F(v)$ of V. The function (2) is differentiable with respect to v over the entire interval in which V takes on values, and the derivative is continuous. Thus, it is absolutely continuous; hence the random variable V has density function, $f(v)$, and this is given by the derivative F' of the distribution function F, i.e., $f(v) = F'(v) = m^k \cdot v^{-(k+1)}$. This function is referred to as the Power Law, and it is usually written in the following general form:

$$f(v) = C \cdot v^{-\alpha}; \quad (3)$$

C is a — problem-dependent — constant, α will be referred to as the exponent of the Power Law. Zipf showed (Zipf, 1949) that the number f of occurrences of words in an english text follows a Power Law with respect to words rank r, i.e., $f(r) = r^{-\alpha}$. Another example is the rank of cities by the number of inhabitants. In general, the Power Law can be used to describe phenomena with frequent small events and rare large events (Adamic, 2003). For visualisation purposes, the Power Law is better represented in a log-log plot, i.e., the straight line obtained by taking the logarithm of (3):

$$\log f(v) = \log C - \alpha \times \log v. \quad (4)$$

log v is represented on the abscissa, log $f(x)$ is represented on the ordinata, $-\alpha$ is the slope of the straight line, and log C is its intercept.

Given a sequence of values $X = (x_1, ..., x_i, ..., x_n)$ on the abscissa, and another sequence of values $Y = (y_1, ..., y_i, ..., y_n)$ on the ordinata. If the correlation coefficient $r(X, Y)$ suggests a fairly strong correlation — it is, say, close to |1| — between X and Y at a log scale, then a regression line can be drawn to exhibit a relationship between the data X and Y; using the slope and the intercept of the regression line the corresponding Power Law can be written. It should be noted, however, that even a good correlation of the two quantities X and Y does not mean that one is necessarily a function of the other; nevertheless, the Power Law can be used as an approximation or estimation of a a behaviour between X and Y, especially when no other relationship is known.

3 POWER LAW OF THE WORLD WIDE WEB

Faloutsos et al. (1999) arrived at the result that, using data provided by the National Laboratory for Applied Networks Research between the end of 1997 and end of 1998, the tail of the frequency distribution of an outdegree — i.e., the number of Internet nodes and Web pages with a given outdegree — is proportional to a Power Law as given by equation (3). Their striking observation was that the value of the exponent was practically almost constant: the values obtained for the Power Law exponent α were 2.15; 2.16; 2.2; 2.48. Barabási et al. (2000) — using 325,729 HTML pages involving 1,469,680 links from the *nd.edu* domain — practically confirmed the earlier results obtained for the values of the exponent by showing that the tail of the frequency distribution of the outdegree (number of URLs to HTML documents from within the given HTML document) followed a Power Law with exponent 2.45; for the indegree distribution the exponent was found to be equal to 2.1. The values obtained earlier for the exponent were also confirmed by Pennock et al. (2002), who found — using 100,000 Web pages selected at random from one billion URLs of Inktomi Corporation Webmap, they binned the frequencies using histograms — that the exponent for the tail of the frequency distribution of the outdegree was 2.72, whereas 2.1 for the case of indegrees.

In order to assess the Power Law behaviour of the degree frequency distribution in the Web we performed the following experiment. Using the Barabási-data[1], we repeated the fitting of a Power Law curve to outdegree distribution. The data was provided as a zipped file; after unzipping it the result was a text file which contained two numbers in each line: the first number was the sequence number of Web pages (0; 1; 2; …; 325,729), the second number was the sequence number of the Web page pointed to by the page represented by the first number. It is interesting to note that the exponent of the Web Power Law is starting to stabilise around the value $\alpha = 2.5$ if the number of Web pages involved is fairly high, above 100,000. For example, for 30,000 pages, the correlation — at a log scale — *r* between outdegree and frequency was only $r = -0.892$, and the fitting of a Power Law curve $C \cdot x^{-\alpha}$ using Mathcad's in-built curve fitting command *genfit* resulted in $\alpha = 0.867$ with an approximation error of the sum of the absolute values of differences of 3.7×10^{6} at 10^{-4} convergence error, whereas using linear regression yielded $\alpha = 1.47$ with an approximation error of 1,589,104 at 10^{-4} convergence error. Figure 1 shows our results for a number of 256,062 Web pages — involving 1,139,426 links — selected at random from the provided 325,729 pages. After processing this file the *X* data consisted of the outdegrees of Web pages, whereas the *Y* data consisted of the corresponding frequencies. For example, there were 2,206 pages having outdegree 13, and the outdegree 14 had its frequency equal to 1,311. The empirical correlation coefficient — taking log scale data — *r* between outdegree and frequency was $r = -0.94$, which indicated a fairly strong stochastic correlation such that the degree frequency *Y* decreased with the outdegree *X*. The linear regression yielded the following values: $\alpha = 2.5$ for the exponent; and $C = 10^{6.1043}$ for the constant. The computation was performed using Matchcad's in-built *line* command; the numeric computation used in this command as well as the fact that we used 69,667 pages less may account for the difference of 0.05 in the exponent value compared to the value obtained by Barabási et al. (see above). Thus, we believe that the difference of 0.05 is not an important one, and that the values obtained in our experiment do confirm the earlier results, further that the Power Law characterises the behaviour of the Web at very large magnitudes.

[1] Provided at http://www.nd.edu/~networks/database/index.html; downloaded January 2, 2004

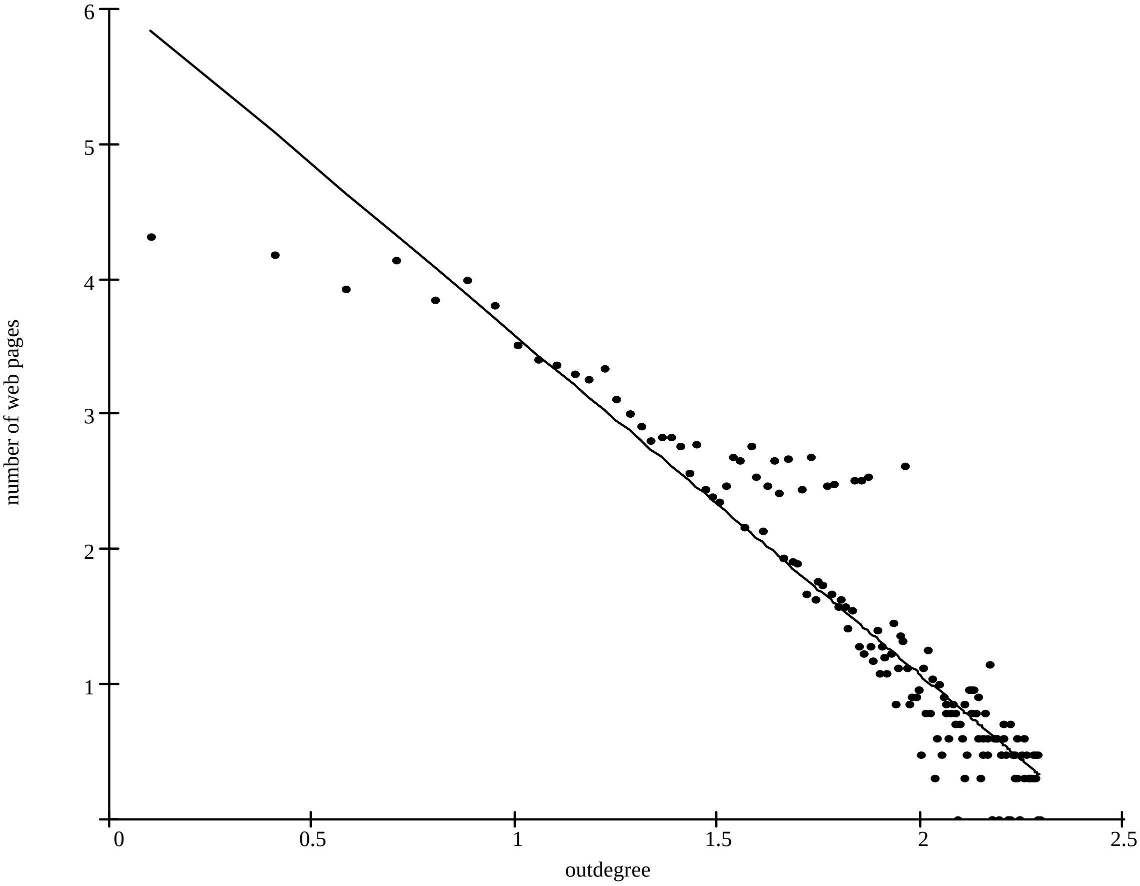

Figure 1. World Wide Web Power Law. The frequency (i.e., number of Web pages) of the outdegrees of Web pages plotted at a log-log scale. The points represent real values, the straight line represents the regression line fitted to the real values. The correlation value is equal to $r = -0.94$, the Power Law exponent is equal to $\alpha = 2.5$.

The different values obtained experimentally for the exponent of the Power Law behaviour of the degree distribution of Web pages and Internet nodes are summarised in Table 1. The values obtained for the exponent have mean $m = 2.316$ with a standard deviation $s = 0.2$. The mean m is larger than $\sqrt{5} = 2.236$ by 0.07, whereas the standard deviation of exponent values relative to $\sqrt{5}$ is equal to 0.21, which is only 0.01 higher than the deviation from the mean.

Table 1. Values obtained for the exponent of the degree Power Law for the World Wide Web.

Faloutsos et al. (1999)				Barabási et al. (2000)		Pennock et al. (2002)		Our own experiment (2004)	Govindan & Tangmunarunkit (Albert, 2000)
2.15	2.16	2.2	2.48	2.45	2.1	2.1	2.72	2.5	2.3

In other words, it may be said that — quite remarkably! — the Web Power Law for the degree frequency distribution of nodes is not just any power law but a very particular one, namely one in which the value of the exponent seems to stabilise around a specific value which can be taken as being equal to $\sqrt{5}$. Taking into account these, it may be assumed that the Power Law for the World Wide Web can be formulated by as follows:

Conjecture (World Wide Web Power Law). The frequency, f, of the degrees, x, in the Web (Web page or Internet node), follows the Power Law $C \cdot x^{-\sqrt{5}}$, i.e.,

$$f \propto C \cdot x^{-\sqrt{5}},$$

where C is a — problem-dependent — constant.

Both from a practical (experimental, applicative, computational) and theoretical (study of properties, formalism) point of view, the right-hand side of the Power Law in the *Conjecture* may be interpreted as representing a curve that is fitted to the actual distribution values, and thus the Power Law can be re-written in the following approximate and useful form:

$$f(x) \approx C \cdot x^{-\sqrt{5}}, \qquad (5)$$

where $f(x)$ denotes — approximate values of — the frequencies of the nodes with degree x. The advantages of this interpretation will be also supported by subsequent experimental and theoretical results. Thus, in what follows, the form (5) of the Web Power Law will be used.

4 GOLDEN SECTION, FIBONACCI AND LUCAS NUMBERS

The Golden Section, the Fibonacci and the Lucas numbers, and the Riemann Zeta-function as well as their properties are well-known in mathematics. In the present paper only those are recalled which are of interest to us in establishing properties of the Web Power Law.

4.1 Golden Section

The Golden Section (*aka* Golden Ratio, Golden Mean, Divine Proportion) is one of the most ancient and overdone yet evergreen topics in mathematics. It is also far-reaching in several other fields, e.g., art, architecture, biology, music, physics. The Golden Section can have as many significances as disciplines in which it appears or is being applied but there is a common agreement that it always relates — subjectively — to a notion of ’beauty’ of the field. For example, it is believed that rectangles whose width-to-height ratio is the Golden Section are the most pleasing to the human eye, or that the timing of musical pieces is considered to be most pleasing to human ears when in Golden Section.

The *Golden Section* is usually denoted by φ, and can be defined in several equivalent ways. In this paper it is defined as the smallest root of the equation $x^2 - x - 1 = 0$; i.e., $\varphi = (\sqrt{5} - 1)/2 \approx 0.61803398875$; the other root is $\Phi = (\sqrt{5} + 1)/2 \approx 1.61803398875$. It is easy to see that the following relationships hold:

$$\sqrt{5} = 2\varphi + 1 \tag{6}$$

$$\varphi \cdot \Phi = 1 \tag{7}$$

Due to the relationships (6) and (7) practically any of φ or Φ may be referred to as the Golden Section.

4.2 Fibonacci Numbers

The *Fibonacci numbers* are defined recursively: $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$, i.e., the sequence of Fibonacci numbers is as follows: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, ... A noteworthy property of the Fibonacci numbers is that the ratio of the consecutive numbers has limit equal to the Golden Section, namely:

$$13/8 = 1.625; \quad 21/13 = 1.615; \quad 34/21 = 1.619; \ldots; \text{ i.e., } \lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \Phi \tag{8}$$

$$5/8 = 0.625; \quad 8/13 = 0.615; \quad 13/21 = 0.619; \; ...; \text{ i.e., } \lim_{n\to\infty} \frac{F_n}{F_{n+1}} = \varphi \qquad (9)$$

The convergence is fairly quick; for example, for $n > 17$ the error is less than 10^{-6}. The Golden Section and the Fibonacci numbers are related by Binet's formula:

$$F_n = \frac{1}{\sqrt{5}}\left(\Phi^n - (-\varphi)^n\right), \qquad (10)$$

from which, and using relationships (6) and (7), it follows that.

$$(-1)^n \cdot \varphi^{2n} + F_n \cdot (2\varphi + 1) \cdot \varphi^n = 1, \quad n = 0, 1, 2, ... \qquad (11)$$

4.3 Lucas Numbers

If the recurrence relation $L_n = L_{n-1} + L_{n-2}$, $n \geq 2$, for the Fibonacci numbers is initialised with the numbers $L_0 = 2$, $L_1 = 1$, then one obtains the sequence 2, 1, 3, 4, 7, 11, 18, 29, 47, 76, ...; the elements of this sequence are known as the *Lucas numbers* L_n. It can be shown, e.g., using induction on n, that the Fibonacci and Lucas numbers are bound by the following relationship:

$$L_n = F_{n-1} + F_{n+1}, \quad n = 1, 2, 3, \; ... \qquad (12)$$

5 PROPERTIES OF THE POWER LAW BEHAVIOUR OF THE WEB

In this part, new, interesting and useful properties of the Web Power Law are established using the mathematical relationships presented in Part 4.

5.1 'Golden' Deviation: 'beauty' in the Web

Taking into account the relationships (5) and (6), the Web Power Law can be written in the following form:

$$f(x) \approx C \cdot x^{-\sqrt{5}} = C \cdot x^{-2\varphi-1} \quad (13)$$

Let us refer to the line γ given by the equation

$$\gamma: \quad Golden\_Line(x) = \log C - \varphi \cdot x \quad (14)$$

having its slope, −φ, equal to the Golden Section in absolute value as the ‘Golden Line’. The Power Law, written as (14), takes the form of a line ω when represented in a log-log drawing; the line ω may thus be referred to as the Web Power Law line, and its equation is as follows:

$$\omega: \quad \log f(x) = \log C - (2\varphi + 1) \cdot \log x \quad (15)$$

Then the angle between the Golden line γ and the Web Power Law line ω has the value

$$\operatorname{atan}\left(\frac{\varphi + 1}{1 + \varphi + 2\varphi^2}\right) \approx 34.187683° \approx 34°,$$

which is approximately equal to the ‘Golden’ angle

$$\operatorname{atan} \varphi = 31.71747441° \approx 32°$$

i.e., to the absolute value of the slope of the Golden Line. Hence, the following concept of a ‘beauty’ in the Web may be formulated (Figure 2):

Property 1 (‘Golden’ Deviation: ‘Beauty’ in Web). The Web Power Law line, ω, deviates by a ‘Golden slope’ from the Golden Line, γ.

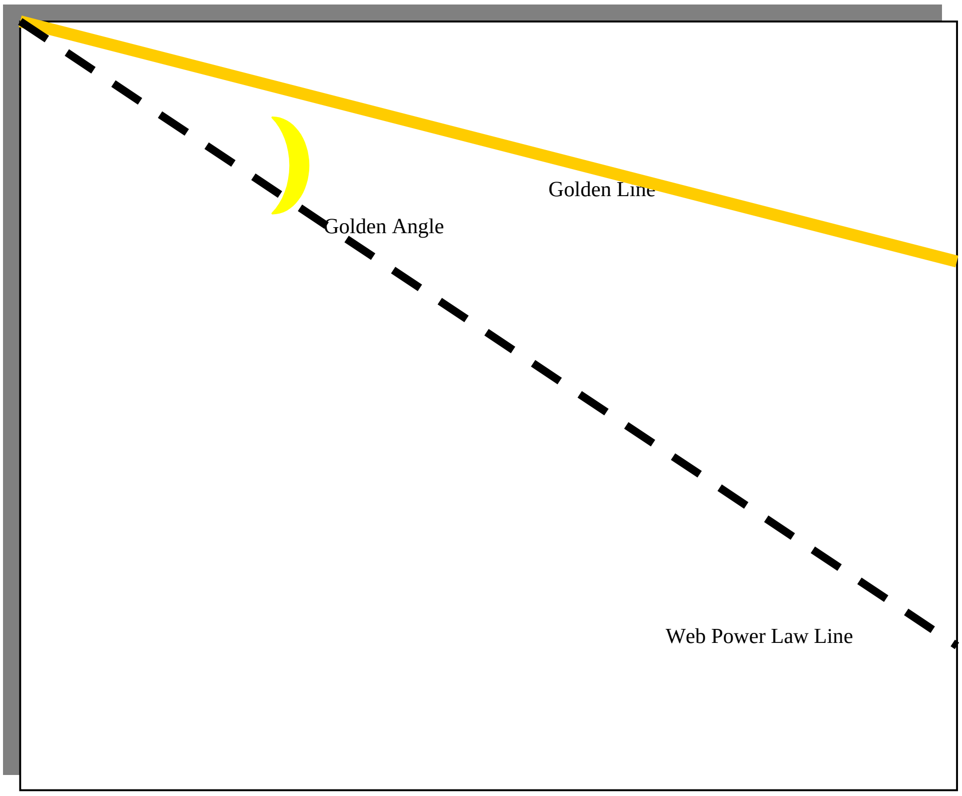

Figure 3. 'Golden' deviation of the Web Power Law from the Golden Line. The solid line represents the Golden Line, i.e., the line whose slope is equal to the Golden Section (in absolute value). The dotted line corresponds to the degree distribution Web Power Law. The measure of the angle between these two lines is aproximately equal to the Golden Section.

5.2 Constructing the Web Power Law using Fibonacci and Lucas numbers

Taking into account the relationship (9), for sufficiently large values of *n*, $2\varphi+1$ can be approximated as follows:

$$2\varphi+1 \approx 2\cdot\frac{F(n-1)}{F(n)}+1=\frac{F(n-1)+F(n-1)+F(n)}{F(n)}, \tag{16}$$

which — given the recursive definition of the Fibonacci numbers, and taking into account the formula (12) — becomes

$$2\varphi+1 \approx \frac{F(n-2)+F(n-3)+F(n-1)+F(n)}{F(n)} =$$

$$= \frac{L(n-1)+L(n-2)}{F(n)} = \frac{L(n)}{F(n)}. \qquad (17)$$

For example, $L(n)/F(n)$ is within an error of 10^{-6} to $2\varphi+1$ for $n \geq 19$. Thus, the Web Power Law (re-writes in a form in which the exponent is expressed using both Fibonacci and Lucas numbers as follows:

$$f(x) \approx C \cdot x^{-\frac{L(n)}{F(n)}} \qquad (18)$$

Taking the logarithm of the relationship (18), one can write the following:

$$\log f(x) \approx \log C - \frac{L(n)}{F(n)} \log x$$

$$\log f(x) + \frac{L(n)}{F(n)} \log x \approx \log C$$

$$F(n) \cdot \log f(x) + L(n) \cdot \log x \approx F(n) \cdot \log C \qquad (19)$$

$$f(x)^{F(n)} \cdot x^{L(n)} \approx C^{F(n)} \qquad (20)$$

Obviously, for a pure Power Law, the relationship (20) holds almost exactly (within an inherent numeric error), as shown in the following example for $f(x) = 5x^{-L(12)/F(12)}$:

$f(x) := 5 \cdot x^{-\sqrt{5}}$ $\quad F(12) = 144$ $\quad L(12) = 322$

$\frac{L(12)}{F(12)} = 2.23611111$ $\quad 5^{F(12)} = 4.48415509 \times 10^{100}$

$f(x) =$

5
1.0613203
0.42864173
0.22528016
0.13678093
0.09098523
0.06445748
0.04781888

$f(x)^{F(12)} \cdot x^{L(12)} =$

$4.48415509 \cdot 10^{100}$
$4.50350235 \cdot 10^{100}$
$4.51485845 \cdot 10^{100}$
$4.52293309 \cdot 10^{100}$
$4.52920622 \cdot 10^{100}$
$4.53433819 \cdot 10^{100}$
$4.53868175 \cdot 10^{100}$
$4.54244767 \cdot 10^{100}$

Naturally, for true Web data, $f(x)$ is not a computed value but the actual frequency, while the Power Law exponent is slightly different from $L(n)/F(n)$. However, it can be shown experimentally that this relationship does hold acceptably well in Web reality.

Let X_k denote the actual page degrees and Y_k denote the corresponding actual frequency ($k = 1, 2, \ldots, M$). Then:

$$F(n) \cdot \log Y_k + L(n) \cdot \log X_k \approx F(n) \cdot \log C \qquad (21)$$

Because the relationship (19) should hold for every $k = 1, 2, \ldots, M$, the mean of the left-hand side taken over all k should equal $F(n) \cdot \log C$ (of course, with an inherent approximation error):

$$\frac{1}{M} \sum_{k=1}^{M} \left(F(n) \cdot \log Y_k + L(n) \cdot \log X_k \right) \approx \frac{1}{M} \sum_{k=1}^{M} F(n) \cdot \log C =$$

$$= \frac{1}{M} \cdot M \cdot F(n) \cdot \log C = F(n) \cdot \log C. \qquad (22)$$

This property makes it possible to formulate the following method for writing a specific Power Law for given real Web data:

Step 1. Establish the number of degrees (e.g., outlinks) X_k (in ascending order) and their corresponding frequencies Y_k, , $k = 1, 2, \ldots, M$, for the Web or Internet nodes under focus.

Step 2. Choose some n, e.g., $n = 8$, 9 or 19, and compute the corresponding Fibonacci number $F(n)$ and Lucas number $L(n)$ using, for example, Binet's formula (10) and formula (12) respectively (or other formulas available).

Step 3. Compute the left-hand side of the relationship (21), i.e.,

$$S_k = F(n) \cdot \log Y_k + L(n) \cdot \log X_k, \quad k = 1, 2, \ldots, M.$$

Step 4. Compute the mean of S_k over all k, i.e.,

$$\mu = \frac{1}{M} \sum_{k=1}^{M} S_k .$$

Step 5. Compute the approximate value for the constant C as follows:

$$C = 10^{\frac{\mu}{F(n)}} .$$

Step 6. Write the specific Power Law for the real Web or Internet portion under focus as follows:

$$f(x) \approx C \cdot x^{-\frac{L(n)}{F(n)}}$$

where, as already seen, x denotes degree and $f(x)$ denotes frequency.

The method given by Steps 1-6 constitutes an alternative method to obtain the Web Power Law (apart from the already known methods of linear regression and curve fitting):

Property 2 (Constructing the Web Power Law). The constant C of the Web Power Law can be approximately calculated using the Steps 1 through 6 with any Fibonacci and Lucas numbers for sufficiently large n.

This method presents a clear computational advantage in terms of complexity compared to the other two methods already known. Both linear regression and curve fitting, even when they are provided as built-in facilities of a software at hand, perform — in the background — numerical optimisation which is, as well known, very computation — and hence also time — demanding. The method proposed by the Steps 1-6 is clearly much more less costly in terms of computation complexity. In order to test both the applicability and the numerical error of the method proposed above the following experiments were performed.

Experiment. Using the data of Experiment 1, we applied the relationship (28) for the following values of $F(n)$ and $L(n)$:

$$F(8) = 21, \quad L(8) = 47; \quad F(19) = 4181, \quad L(19) = 9349; \quad (23)$$
$$F(22) = 17711, \quad L(22) = 39603; \quad F(29) = 514229, \quad L(29) = 1149851.$$

The numeric values obtained for the left-hand side oscillate around the value obtained for the right-hand side The corresponding means for different values of n are as follows: 112; 22,332; 94,603; 274,6752, respectively; these values are approximately equal to the corresponding right-hand sides; the corresponding average errors (taken as the mean of the absolute values of differences) are as follows: 17; 3,451; 1,462; 424,506. The Steps 1–6 result in the following Power Laws:

$$f(x) = 10^{\frac{126}{21}} \cdot x^{-2.23} = 969321 \cdot x^{-2.23}$$
$$f(x) = 10^{\frac{25013}{4181}} \cdot x^{-2.23} = 960454 \cdot x^{-2.23}$$
$$f(x) = 10^{\frac{105956}{17711}} \cdot x^{-2.23} = 960454 \cdot x^{-2.23}$$
$$f(x) = 10^{\frac{3076363}{514229}} \cdot x^{-2.23} = 960454 \cdot x^{-2.23}$$

It can be seen that the values obtained for the constant are fairly stable and compare well with that obtained earlier in Experiment 1 using linear regression ($\approx 10^6$).

Using the data from the experiment, the Power Law for the rank of indegrees is as follows (obtained by linear regression):

$$f(x) = 10^{5.4} \cdot x^{-2.2} \approx 10^{5.4} \cdot x^{-2\varphi-1},$$

with correlation $r = -0.94$ (Figure 4).

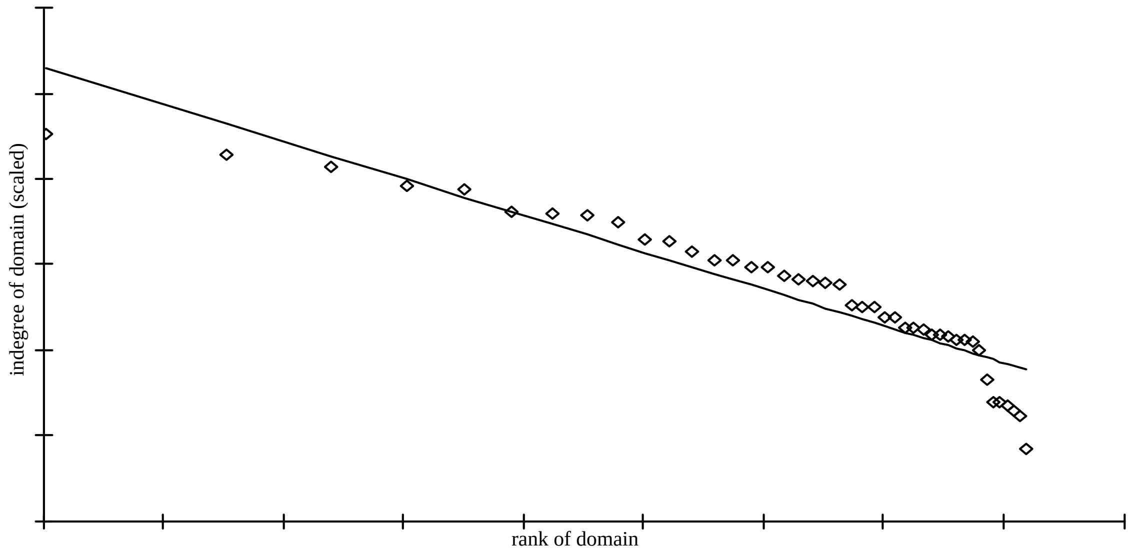

Figure 4. Rank of the indegree of country domain names (log-log drawing). The diamonds represent actual data, the line represents the regression line. The correlation is equal to $r = -0.94$, the power law is $f(x) = 10^{5.4} \cdot x^{-2.2}$.

Using the Fibonacci number 514,229 and the corresponding Lucas number 1,149,851 the mean for the real data is equal to μ = 277,306, and thus the constant is equal to $C = 10^{277306/514229}$ = 246,976, which compares fairly well with that obtained by linear regression ($10^{5.4}$ = 251,187).

6 CONCLUSIONS

In this paper, based on experimental results obtained in the last six or seven years as well as on own experimental results as reported in this paper, it was conjectured that the power law characterising the distribution of degrees in the Web and Internet is not just any power law but a very specific one, namely one having a very specific exponent: $\sqrt{5}$. It was then shown that the log-log graph of the Web Power Law for nodes (Web pages, Internet nodes) is a straight line that deviates from the straight line having slope equal to the Golden Section by an angle whose measure is approximately equal to the Golden Section (Property 1). Because the presence of the Golden Section in a discipline has always been traditionally considered as an expression of a beauty, this property may be seen as an expression of a balance or proportion in a law governing the evolution of Web link topology.

A computation method was given, and supported by experimental evidence, as regards the construction of the specific Web Power Law for an actual Web portion under focus. This method offers a computationally cheaper alternative to existing methods (linear regression, curve fitting), and it is particularly interesting in that it uses any, sufficiently large, Fibonacci and Lucas numbers. The computationally useful and mathematically interesting relationship between degree frequencies, Golden Section, Fibonacci and Lucas numbers contained in this property can open up further possibilities to involve number theory into the computational study of Web topology.

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